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On the Numerical Solution of Time-Dependent Viscous Incompressible Fluid Flows Involving Solid Boundaries

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An inherent numerical problem associated with the fully explicit pseudospectral numerical simulation of the incompressible Navier–Stokes equation for viscous flows with no-slip walls is described. A semi-implicit scheme which circumvents this numerical difficulty is presented. In this algorithm the equation of continuity rather than the Poisson equation for pressure is solved directly. Pseudospectral formulation of the channel flow problem using Fourier series and Chebyshev polynominals expansions is given for this scheme. An example demonstrating the applicability of the method is given.

1. INTRODUCTION

With the advent of large and fast computers in recent years, the three-dimensional time-dependent computation of turbulent flows is becoming feasible. One of the promising approaches for turbulent flow calculations is the large eddy simulation technique (e.g., [1–5]). In large eddy simulation technique, one numerically solves for the large-scale flow field using the three-dimensional time-dependent Navier–Stokes equations and models the small-scale (subgrid scale) flow field. Both the large eddy simulation technique and the direct simulation approach [6–9] are in their primitive stages and have not been applied to complex flows.

A common objective of the large eddy and direct simulation techniques is to test and suggest statistical models of turbulence which can in turn be used in a simpler method for complex flows. In this case, it is imperative that errors introduced by the numerical scheme are minimized. This is necessary for an objective evaluation of the turbulence model.

Spectral or pseudospectral methods [10] have a very high formal order of accuracy and are especially well-suited for minimizing numerical errors. In particular with these methods, derivatives are evaluated without phase error. This is not the case with finite-difference or finite-element methods. However, in contrast to spectral approximation, pseudospectral methods are susceptible to aliasing errors. Nevertheless, aliasing is not usually considered to be a serious problem because the damping associated with most physical problems limits aliasing errors and instabilities [11]. In any case, aliasing errors can formally be eliminated by an appropriate filtering technique [6, 9].

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For problems in which only periodic boundary conditions are used (e.g., decay of homogeneous isotropic turbulence) the application of spectral methods is relatively straightforward [3, 6]. For viscous flow problems with no-slip walls, however, one may encounter a serious numerical difficulty with spectral methods. In this paper, we first analyze a numerical problem associated with fully explicit (time advancing) simulation of the Navier–Stokes equations for viscous flows with no-slip walls in which pseudospectral methods are used. We will show that the application of a combined explicit (time advancing) and pseudospectral method to these flows results in nonconvergent series expansions for dependent variables which in turn yield an erroneous computation. Then we present a semi-implicit algorithm which circumvents this problem and gives a pseudospectral formulation of this algorithm for channel flow. In this formulation, flow variables are expanded into Fourier series in the homogeneous directions (x_1 and x_3) and Chebyshev polynomials in the direction normal to the no-slip walls (x_2). Finally, a simple example which has essentially all the mathematical features of turbulent channel flow is worked out.

2. Fundamental Numerical Problem

In this section we describe an inherent numerical difficulty associated with the fully explicit solutions of the Navier–Stokes equations in their primitive form. Consider the momentum equations

$$\frac{\partial u_i}{\partial t} = -\frac{\partial p}{\partial x_i} + H_i, \tag{2.1}$$

where

$$H_i = -\frac{\partial}{\partial x_j} u_i u_j + \nu \frac{\partial^2 u_i}{\partial x_j \, \partial x_j}.$$

In the fully explicit (time advancing) solution of Eq. (2.1), one normally specifies an arbitrary initial solenoidal velocity field satisfying the no-slip condition. Then one proceeds to solve the appropriate Poisson equation for pressure obtained from the application of the divergence operator to the momentum equations to ensure that $\nabla \cdot \mathbf{u} = 0$ at the next time step. The resulting pressure is then used together with the computed H_i in (2.1) to advance u_i in time. At the solid boundaries the Neumann boundary condition,

$$\frac{\partial p}{\partial \mathbf{n}} = \nu \mathbf{n} \cdot \nabla^2 \mathbf{u}, \tag{2.2}$$

is normally used in conjunction with the Poisson equation for pressure. Here $\mathbf{n}$ is a unit vector normal to the solid boundary. This condition is obtained from the normal momentum equation using the no-slip condition.

In addition to the Neumann boundary condition (2.2) for pressure, one can obtain the Dirichlet boundary condition,

$$\frac{\partial p}{\partial \boldsymbol{\tau}} = \nu \boldsymbol{\tau} \cdot \nabla^2 \mathbf{u}, \tag{2.3}$$

from the evaluation of the tangential momentum equations at the wall. Here $\boldsymbol{\tau}$ is a unit vector tangent to the solid boundary. In general, however, the Neumann and Dirichlet problems for pressure may not have the same solution [5, 10]. Only one of the Eqs. (2.2) or (2.3) is sufficient to solve the Poisson equation. Hence, in an explicit time advancing of Eq. (2.1), Eqs. (2.2) and (2.3) are not in general enforced simultaneously. This can cause a serious numerical difficulty if pseudospectral techniques are used to evaluate spatial derivatives. To make this clear, we consider, as an example, three-dimensional time-dependent flow between two parallel plates. The no-slip boundary condition is used at the walls ($x_2 = \pm 1$) and periodic boundary conditions are employed in the streamwise, x_1, and spanwise, x_3, directions.

Chebyshev polynomials (see Section 4) and Fourier series expansions will be used to represent the flow variables in the vertical and horizontal directions, respectively. For the purpose of this section, the use of Chebyshev polynominals is not essential, and any set of orthogonal functions would do. Let

$$p = \sum_{m=0}^{N_2}{}'' \sum_{k_1} \sum_{k_3} a_m(k_1, k_3)\, T_m(x_2)\, e^{i(k_1 x_1 + k_3 x_3)}, \tag{2.4a}$$

$$u_i = \sum_{m=0}^{N_2}{}'' \sum_{k_1} \sum_{k_3} b_{i_m}(k_1, k_3)\, T_m(x_2)\, e^{i(k_1 x_1 + k_3 x_3)}, \tag{2.4b}$$

$$H_i = \sum_{m=0}^{N_2}{}'' \sum_{k_1} \sum_{k_3} c_{i_m}(k_1, k_3)\, T_m(x_2)\, e^{i(k_1 x_1 + k_3 x_3)}, \tag{2.4c}$$

and

$$\frac{\partial p}{\partial x_2} = \sum_{m=0}^{N_2}{}'' \sum_{k_1} \sum_{k_3} a'_m(k_1, k_3)\, T_m(x_2)\, e^{i(k_1 x_1 + k_3 x_3)}, \tag{2.4d}$$

where $T_m(x_2)$ is the mth-order Chebyshev polynomials of the first kind. Double primes indicate that the first and last terms in the series are to be taken with factor 1/2, and $(N_2 + 1)$ grid points are used in the x_2 direction. From (2.4b) we readily obtain

$$\frac{\partial b_{i_m}}{\partial t} = \frac{2}{N_1 N_2 N_3} \sum_{j=1}^{N_2-1} \sum_{x_1} \sum_{x_3} \frac{\partial u_i}{\partial t} \cos m\theta_j e^{-i(k_1 x_1 + k_3 x_3)}, \tag{2.5}$$

where $x_{2_j} = \cos \theta_j$ (see Section 4), $\theta_j = \pi j / N_2$, $j = 0, 1, 2, \ldots, N_2$, and N_1 and N_3

are the number of mesh points in x_1 and x_3 directions, respectively. Note that here we have enforced the no-slip boundary conditions, that is,

$$\left.\frac{\partial u_i(x_1, \theta_j, x_3)}{\partial t}\right|_{j=0,N_2} = 0.$$

Substituting the right-hand side of (2.1) for $\partial u_i/\partial t$ in (2.5) and using (2.4) together with the orthogonality of the expansion functions, we obtain

$$\frac{\partial b_{l_n}}{\partial t} = \begin{pmatrix} -ik_1 a_n(k_1, k_3) \\ -a_n'(k_1, k_3) \\ -ik_3 a_n(k_1, k_3) \end{pmatrix} + c_{l_n} - \frac{1}{N_2} \sum_{m=0}^{N_2}{}'' \left\{ \begin{pmatrix} -ik_1 a_m(k_1, k_3) \\ -a_m'(k_1, k_3) \\ -ik_3 a_m(k_1, k_3) \end{pmatrix} + c_{l_m} \right\} \times \{(-1)^{n+m} + 1\}. \tag{2.6}$$

The last term in (2.6), which is the result of enforcing the no-slip boundary conditions, is the source of trouble. In general, it has a finite value irrespective of the value of n, and, hence, does not approach a vanishing level for arbitrary large values of n (or N_2). This in turn results in a nonconvergent series expansion for $\partial u_i/\partial t$ which in practice yields an erroneous and meaningless computation. It should be noted that the last term in (2.6) is identically equal to zero if (2.2) and (2.3) are enforced simultaneously. However, as was mentioned earlier, only one of them is used as the boundary condition for the Poisson equation.

Before concluding this section, we emphasize that the problem addressed here can be *avoided* if one uses three-point finite differences rather than spectral techniques to approximate partial derivatives in the direction normal to the solid boundaries [5]. But this problem does cause serious numerical difficulty if spectral methods are used.

3. A Semi-Implicit Scheme

The roots of the numerical difficulty discussed in the previous section lie in the fact that the continuity equation is enforced indirectly through a Poisson equation which has nonunique boundary conditions and the fact that explicit (time advancing) schemes treat the problem as an initial value problem rather than an initial-boundary value problem.

In what follows we outline a method that circumvents the numerical difficulty mentioned in the previous section. This numerical scheme treats the pressure and viscous terms in the Navier–Stokes equation implicitly and the remaining terms explicitly. The equation of continuity rather than the Poisson equation for pressure is solved directly.

Let us start with the Navier–Stokes equations:

$$\frac{\partial u_i}{\partial t} = \mathscr{L}_i - \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \, \partial x_j}, \tag{3.1}$$

where $\mathscr{L}_i$ is equal to the convective terms. For time advancing we use the second-order Adams–Bashforth method [12] on $\mathscr{L}_i$ and the Crank–Nicolson method [13] on $\partial p/\partial x_i$ and $\partial^2 u_i/\partial x_j\,\partial x_j$. Thus we have

$$u_i^{n+1} = u_i^n + \frac{\Delta t}{2}(3\mathscr{L}_i^n - \mathscr{L}_i^{n-1}) - \frac{\Delta t}{2}\left(\frac{\partial p^{n+1}}{\partial x_i} + \frac{\partial p^n}{\partial x_i}\right) + \nu\frac{\Delta t}{2}\left(\frac{\partial^2 u_i^{n+1}}{\partial x_j\,\partial x_j} + \frac{\partial^2 u_i^n}{\partial x_j\,\partial x_j}\right), \tag{3.2}$$

where superscripts denote the time step. Let $\beta = 2/\nu\,\Delta t$. Rearrangement of (3.2) yields

$$\frac{\partial^2 u_i^{n+1}}{\partial x_j\,\partial x_j} + \beta u_i^{n+1} + \beta\frac{\Delta t}{2}\frac{\partial p^{n+1}}{\partial x_i} = \beta u_i^n + \beta\frac{\Delta t}{2}(3\mathscr{L}_i^n - \mathscr{L}_i^{n-1}) - \beta\frac{\Delta t}{2}\frac{\partial p^n}{\partial x_i} - \frac{\partial^2 u_i^n}{\partial x_j\,\partial x_j}. \tag{3.3}$$

Finally, we write the continuity equation at time step $n+1$

$$\frac{\partial u_i^{n+1}}{\partial x_i} = 0. \tag{3.4}$$

For the channel flow (see Section 2), let us Fourier transform (3.3) and (3.4) in the streamwise, x_1, and the spanwise, x_3, directions. This transformation converts the set of partial differential equations (3.3) and (3.4) to a set of ordinary differential equations for every pair of Fourier modes k_1 and k_3 with x_2 as the independent variable. Note that for flows that are homogeneous in streamwise and spanwise directions (e.g., fully developed turbulent channel flow, plane Couette flow), periodic boundary conditions can be used in these directions. The use of periodic boundary conditions in the homogeneous directions can be justified if the lengths of the computational box are large enough to include the important large eddies [14]. In the remainder of this section all the variables with a superscript are to be interpreted as two-dimensional, Fourier-transformed quantities. Fourier transforming (3.3) and (3.4) results in the following set of ordinary differential equations for the dependent variables

$$\frac{d^2 u_1^{n+1}}{dx_2^2} + (\beta - k_1^2 - k_3^2)\,u_1^{n+1} + ik_1\beta\frac{\Delta t}{2}p^{n+1} = (\beta - k_1^2 - k_3^2)\,u_1^n + \beta\frac{\Delta t}{2}(3\mathscr{L}_1^n - \mathscr{L}_1^{n-1}) - ik_1\beta\frac{\Delta t}{2}p^n - \frac{d^2 u_1^n}{dx_2^2}, \tag{3.5a}$$

$$\frac{d^2 u_2^{n+1}}{dx_2^2} + (\beta - k_1^2 - k_3^2)\,u_2^{n+1} + \beta\frac{\Delta t}{2}\frac{dp^{n+1}}{dx_2} = (\beta - k_1^2 - k_3^2)\,u_2^n + \beta\frac{\Delta t}{2}(3\mathscr{L}_2^n - \mathscr{L}_2^{n-1}) - \beta\frac{\Delta t}{2}\frac{\partial p^n}{\partial x_2} - \frac{d^2 u_2^n}{dx_2^2}, \tag{3.5b}$$

$$\frac{d^2u_3^{n+1}}{dx_2^2} + (\beta - k_1^2 - k_3^2)\, u_3^{n+1} + ik_3\beta \frac{\Delta t}{2} p^{n+1}$$
$$= (\beta - k_1^2 - k_3^2)\, u_3^{n} + \beta \frac{\Delta t}{2} (3\mathscr{L}_3^n - \mathscr{L}_3^{n-1}) - ik_3\beta \frac{\Delta t}{2} p^{n} - \frac{d^2u_3^{n}}{dx_2^2}, \tag{3.5c}$$

and

$$ik_1u_1^{n+1} + \frac{du_2^{n+1}}{dx_2} + ik_3u_3^{n+1} = 0, \tag{3.5d}$$

where

$$i = \sqrt{-1}.$$

Thus, for every pair of k_1 and k_3, we have four coupled linear ordinary differential equations with $u_1^{n+1}(k_1, x_2, k_3)$, $u_2^{n+1}(k_1, x_2, k_3)$, $u_3^{n+1}(k_1, x_2, k_3)$, and $p^{n+1}(k_1, x_2, k_3)$ as unknowns. In the next section we will describe the solution technique for the above set of ordinary differential equations using the Chebyshev polynomials.

4. Solution Method Using Chebysev Polynomials

Equations (3.5a)–(3.5d) have successfully been solved for the case of turbulent channel flow using second-order finite difference schemes to approximate derivatives in the vertical, x_2, direction [5]. In this section, we use Chebyshev polynomials to approximate derivatives in the x_2 direction. The resulting system of equations is then solved for the Chebyshev–Fourier series expansion coefficients. The essential feature of the Chebyshev polynomial approximation is that it has infinite order accuracy (in the sense that for an infinitely differentiable function, errors decrease more rapidly than any power of $1/N$ as $N \to \infty$) and can be implemented very efficiently using fast Fourier transform routines [10]. The powerful convergence of Chebyshev series has been used by Lanczos [15] in numerical solution of differential equations as far back as four decades ago. Properties of Chebyshev polynomials can be found in Lanczos [16], Fox and Parker [17], etc. An important difference between Fourier series expansion and Chebyshev series expansion lies in the capability of handling boundary conditions. The latter requires no special boundary conditions in contrast to the former which requires periodic boundary conditions. For nonperiodic boundary conditions the well-known Gibbs phenomenon near the boundary destroys the convergence rate of Fourier series expansion. For this reason, in the numerical simulation of fluid–flow problems involving no-slip boundary condition, Chebyshev polynomials are preferred to any other sets of orthogonal functions [7, 10]. In addition, note that $T_m(\cos\theta) = \cos m\theta$. Thus, the transformation ($x_2 = \cos\theta$), which is adequate for boundary layer coordinate stretching, renders the evaluation of the Chebyshev expansion coefficients particularly efficient with the use of FFT routines.

We begin by expanding the two-dimensional Fourier transform of flow variables in terms of Chebyshev polynomials. Let

$$u_1^{n+1}(k_1, x_2, k_3) = \sum_{m=0}^{N_2}{}'' a_m(k_1, k_3)\, T_m(x_2),$$

$$u_2^{n+1}(k_1, x_2, k_3) = \sum_{m=0}^{N_2}{}'' b_m(k_1, k_3)\, T_m(x_2),$$

$$u_3^{n+1}(k_1, x_2, k_3) = \sum_{m=0}^{N_2}{}'' c_m(k_1, k_3)\, T_m(x_2), \tag{4.1}$$

$$p^{n+1}(k_1, x_2, k_3) = \sum_{m=0}^{N_2}{}'' d_m(k_1, k_3)\, T_m(x_2),$$

$$\frac{du_1^{n+1}}{dx_2}(k_1, x_2, k_3) = \sum_{m=0}^{N_2}{}'' a'_m(k_1, k_3)\, T_m(x_2),$$

and

$$\frac{d^2u_1^{n+1}}{dx_2^2}(k_1, x_2, k_3) = \sum_{m=0}^{N_2}{}'' a''_m(k_1, k_3)\, T_m(x_2).$$

Similar relations hold for derivatives of u_2^{n+1}, u_3^{n+1}, and p^{n+1}. Note also that a'_{N_2}, a''_{N_2}, and a''_{N_2-1} are identically zero.

Substituting (4.1) into (3.5a) we obtain

$$a''_m + (\beta - k_1^2 - k_3^2)\, a_m + ik_1\beta \frac{\Delta t}{2} d_m = q_{1_m}, \qquad m = 0, 1, 2,..., N_2 \tag{4.2}$$

where q_{1_m}'s are the coefficients of Chebyshev expansion of the right-hand side of Eq. (3.5a). Using the recursion formulas between a'_m, a''_m, and a_m (see [17]),

$$\begin{aligned} 2ma_m &= a'_{m-1} - a'_{m+1}, \\ 2ma'_m &= a''_{m-1} - a''_{m+1}, \end{aligned} \tag{4.3}$$

we obtain

$$\begin{aligned} 2ma'_m + (\beta - k_1^2 - k_3^2)(a_{m-1} - a_{m+1}) + ik_1\beta \frac{\Delta t}{2} (d_{m-1} - d_{m+1}) \\ = q_{1_{m-1}} - q_{1_{m+1}}, \qquad m = 1, 2,..., N_2 + 1. \end{aligned} \tag{4.4}$$

A further application of Eq. (4.3) on Eq. (4.4) gives

$$\frac{\beta'}{2(m-1)} a_{m-2} + \left\{2m - \frac{\beta'}{2(m-1)} - \frac{\beta'}{2(m+1)}\right\} a_m + \frac{\beta'}{2(m+1)} a_{m+2}$$

$$+ ik_1\beta \frac{\Delta t}{2} \left[\frac{1}{2(m-1)} d_{m-2} - \left\{\frac{1}{2(m-1)} + \frac{1}{2(m+1)}\right\} d_m + \frac{1}{2(m+1)} d_{m+2}\right]$$

$$= \frac{q_{1_{m-2}} - q_{1_m}}{2(m-1)} - \frac{q_{1_m} - q_{1_{m+2}}}{2(m+1)}, \qquad m = 2,\ldots, N_2 \tag{4.5a}$$

(note that $a_{N_2+1} = a'_{N_2} = a''_{N_2} = a''_{N_2-1} = 0$) where $\beta' = \beta - k_1^2 - k_3^2$. Similar operations to (3.5b)–(3.5d) yield

$$\frac{\beta'}{2(m-1)} b_{m-2} + \left\{2m - \frac{\beta'}{2(m-1)} - \frac{\beta'}{2(m+1)}\right\} b_m$$

$$+ \frac{\beta'}{2(m+1)} b_{m+2} + \beta \frac{\Delta t}{2} (d_{m-1} - \gamma d_{m+1})$$

$$= \frac{q_{2_{m-2}} - q_{2_m}}{2(m-1)} - \frac{q_{2_m} - q_{2_{m+2}}}{2(m+1)}, \qquad m = 2,\ldots, N_2 \tag{4.5b}$$

where

$$\gamma = \begin{cases} 1/2 & \text{for } m = N_2 - 1 \\ 1 & \text{for } m \neq N_2 - 1, \end{cases}$$

$$\frac{\beta'}{2(m-1)} c_{m-2} + \left\{2m - \frac{\beta'}{2(m-1)} - \frac{\beta'}{2(m+1)}\right\} c_m + \frac{\beta'}{2(m+1)} c_{m+2}$$

$$+ ik_3\beta \frac{\Delta t}{2} \left[\frac{1}{2(m-1)} d_{m-2} - \left\{\frac{1}{2(m-1)} + \frac{1}{2(m+1)}\right\} d_m + \frac{1}{2(m+1)} d_{m+2}\right]$$

$$= \frac{q_{3_{m-2}} - q_{3_m}}{2(m-1)} - \frac{q_{3_m} - q_{3_{m+2}}}{2(m+1)}, \qquad m = 2,\ldots, N_2\,, \tag{4.5c}$$

$$ik_1(a_{m-1} - a_{m+1}) + (2 - \delta_{nN_2})\, m b_m + ik_3(c_{m-1} - c_{m+1}) = 0,$$
$$m = 1,\ldots, N_2 + 1. \tag{4.5d}$$

No-slip boundary conditions at the walls ($x_2 = \pm 1$) give

$$\sum_{m=0}^{N_2}{}'' a_m = \sum_{m=0}^{N_2}{}'' (-1)^m a_m = 0,$$

$$\sum_{m=0}^{N_2}{}'' b_m = \sum_{m=0}^{N_2}{}'' (-1)^m b_m = 0, \tag{4.5e}$$

$$\sum_{m=0}^{N_2}{}'' c_m = \sum_{m=0}^{N_2}{}'' (-1)^m c_m = 0.$$

Equations (4.5a)–(4.5d) together with boundary conditions (4.5e) form $4(N_2 + 1)$ equations for $4(N_2 + 1)$ unknowns, thus closing the system *without* any pressure boundary condition at the walls. Now, we can solve for a_m, b_m, c_m, and d_m given q_{i_m} ($i = 1, 2, 3$) which are obtained from the data at the previous time steps ($n, n - 1$). Inverse Fourier and Chebyshev transforms yield u_1^{n+1}, u_2^{n+1}, u_3^{n+1}, and p^{n+1} in the physical space.

It is important to appreciate that in the present numerical formulation all of the governing equations are satisfied at the boundaries as well as inside the domain. Thus, Eqs. (2.2) and (2.3) are simultaneously enforced in this method and the difficulty discussed in Section 2 is absent. In addition, the present formulation implies that in spite of the appearance of the pressure derivatives in Navier–Stokes equations, no pressure boundary conditions are required at the walls. Hence, in the viscous incompressible flow problems only velocity boundary conditions are necessary. This is in accord with the fact that physics does not provide a priori boundary conditions for pressure as it does for velocities.

5. Numerical Example

Due to the simplicity of its geometry and some experimental advantages, channel flow has been a particularly attractive reference flow for both theoretical and experimental turbulence research. As a result, there is a considerable number of experimental as well as theoretical findings available for a detailed evaluation of large eddy or direct simulation techniques. For the purpose of this paper, we give a simple example to demonstrate the applicability of the method described in the previous sections. Although the problem is very simple in nature, it contains all the essential mathematical features of turbulent channel flow as well as basic transitional and turbulent shear flows.

The problem deals with an arbitrary initial velocity field developing under the action of a mean pressure gradient in a two-dimensional channel until the steady-state solution is obtained. The initial velocity field satisfying the continuity equation and no-slip boundary condition is formed with three-dimensional disturbances on top of a two-dimensional mean flow as follows:

$$u_1(x_1, x_2, x_3) = C(1 - x_2^8) + \epsilon \frac{L_1}{2} \sin \pi x_2 \cos \frac{4\pi x_1}{L_1} \sin \frac{2\pi x_3}{L_3},$$

$$u_2(x_1, x_2, x_3) = -\epsilon(1 + \cos \pi x_2) \sin \frac{4\pi x_1}{L_1} \sin \frac{2\pi x_3}{L_3},$$

and

$$u_3(x_1, x_2, x_3) = -\epsilon \frac{L_3}{2} \sin \frac{4\pi x_1}{L_1} \sin \pi x_2 \cos \frac{2\pi x_3}{L_3},$$

where L_1 and L_3 are the lengths of the computational box in the x_1 and x_3 directions, respectively, and ϵ is 10 % of the averaged mean velocity. Note that all the length scales

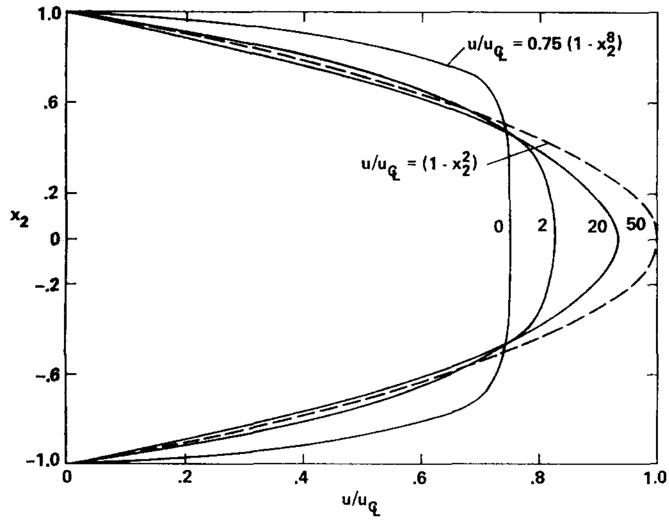

FIG. 1. Results of the computation $x_{2_j} = \cos(\pi j/8)$, $j = 0, 1, \ldots, 8$. u/u_{C_L} is the velocity nondimensionalized by the steady-state centerline velocity; x_2 is the coordinate normal to the wall, nondimensionalized by the channel half width; $u_{C_L}t/h$ is the nondimensional time based on the centerline velocity and the channel half width; and h is the channel half width.

are nondimensionalized by the channel half width to give $x_2 = \pm 1$ at the wall locations. The constant C was chosen such that the initial mass flux would be the same as the final equilibrium mass flux for a given Reynolds number. The steady-state solution can be arrived at irrespective of the values of C, but with this choice one can closely simulate the entry length problem. For the Reynolds number under consideration, Re = 100 (based on averaged mean velocity and the channel width), which is small enough to keep the flow laminar, the three-dimensional disturbances gradually die out and the mean flow develops to a fully developed laminar Poiseulle flow.

For this example, 16 uniform grid points were used in the x_1 direction, 8 uniform grid points in the x_3 direction, and 9 grid points with nonuniform spacings,

$$x_{2_j} = \cos(\pi j/8), \qquad j = 0, 1, \ldots, 8, \tag{5.1}$$

were used in the x_2 direction. Results of the computation are shown in Fig. 1. Velocity profiles are analogous to those of the entry region of channel flow. However, the present profiles are developing in time rather than in space, since periodic boundary conditions are used in the streamwise direction. In addition, during the course of integration the perturbations on the mean flow were damped to a vanishing level.

6. Summary and Conclusion

A fundamental numerical difficulty associated with fully explicit (time advancing) integration of Navier–Stokes equations for incompressible viscous flow problems with no-slip walls is discussed in detail (Section 2). It was shown that, in general, numerical methods that use explicit time advancing together with pseudospectral methods yield nonconvergent series expansions for the dependent variables. A semi-implicit algorithm, which solves the continuity equation directly rather than the Poisson equation for pressure, is presented as a remedy for the difficulty. In doing so, the ambiguity in imposing the proper boundary condition for the Poisson equation is eliminated. A pseudospectral formulation for this algorithm is given. In this formulation, flow variables are expanded into the Fourier–Chebyshev polynomial series. The resulting system of equations is closed without any pressure boundary conditions on the walls.

Based on the discussion of Sections 2 and 4, we believe that in the numerical simulation of viscous incompressible flows with no-slip walls where spectral techniques are to be used, the continuity equation rather than the Poisson equation for pressure should be solved. In addition the numerical method should treat the problem as an initial boundary value problem by treating pressure and viscous terms implicltly.

In a future article, we will apply the numerical method presented in this paper to the case of turbulent channel flow. There a comparison will be made between the results of the present solution procedure and that of [5], where second-order, finite-difference schemes (rather than Chebyshev polynomials) were used to calculate derivatives in the x_2 direction.

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